# ME196 Report: Vehicle Dynamics with Rear Steering

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# 1 Introduction

Accurate and reliable vehicle models are crucial in autonomous vehicles. One area that is commonly overlooked is the vehicle dynamics of a car driven in reverse. Many reverse steering controllers make the assumption that the speed of a car in reverse is at low speed with no side slip of the tires [1]. For example this model that was used for a parallel parking controller at low speeds.

$$\dot{X} = V_x \cos(\psi) \tag{1.1}$$

$$\dot{Y} = V_x \sin(\psi) \tag{1.2}$$

$$\dot{\psi} = -\frac{V_x \tan(\delta)}{l_f + l_r} \tag{1.3}$$

Eq. 1 a kinematic model of the vehicle motion while it is driven backwards [1].

By using existing methods of vehicle modeling, a rear steering model will be created. A common model known as the bicycle model (a simplification that assumes that the left and right tire are one) will be used for the derivation as it accurately depicts a vehicles' behavior with simpler math and geometry. This will also allow for knowledge of the stable operating regions of a vehicle in reverse. A model that will accurately allow for the vehicle characteristics and dynamics calculations is key for applying control algorithms. This will allow future controllers to take advantage of this new model for more robust model based control systems in future autonomous vehicles.

# 2 Kinematic Rear Steering Bicycle Model

The first main difference between the front and rear steering model is the angles of the tires and lateral direction of the vehicle. To understand the changes in the front versus rear steering a direct comparison will be made. When a vehicle is in reverse (steering in the rear of the vehicle)

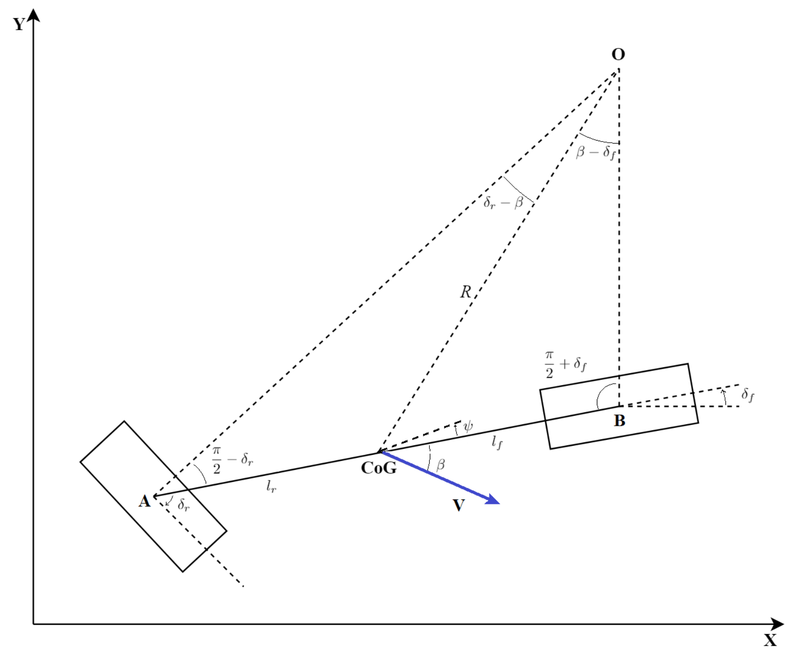


Figure 1: Kinematics of lateral vehicle motion for rear steering bicycle model

Steering for a vehicle can be thought of as a shopping cart. With a front steering vehicle the turning is done by pulling the non-steering rear tires towards the course of the front steered tires. With rear steering the turning is done by pushing the non-steering front tires. This is also the source of the instability of cars in reverse, causing the car's front end to spin around and face the forward direction. This characteristic of rear steering is also how a j-turn is achieved.

Using the same method of derivation as Rajamani's bicycle model [2, p. 27] a rear steering model can be rewritten into a rear steering bicycle model by adjusting the kinematics to a rear steering bicycle model figure 1. Two triangles **OACoG** and **OBCoG** in figure 1 enables solving for $\dot{X}$, $\dot{Y}$, and $\dot{\psi}$ with a kinematic model. A key difference between forward and reverse motion is the behavior of the how the rear steering effects the kinematic model. The rear steering has a velocity vector that is facing outwards of the turn, while the front steering is facing in towards the turn.

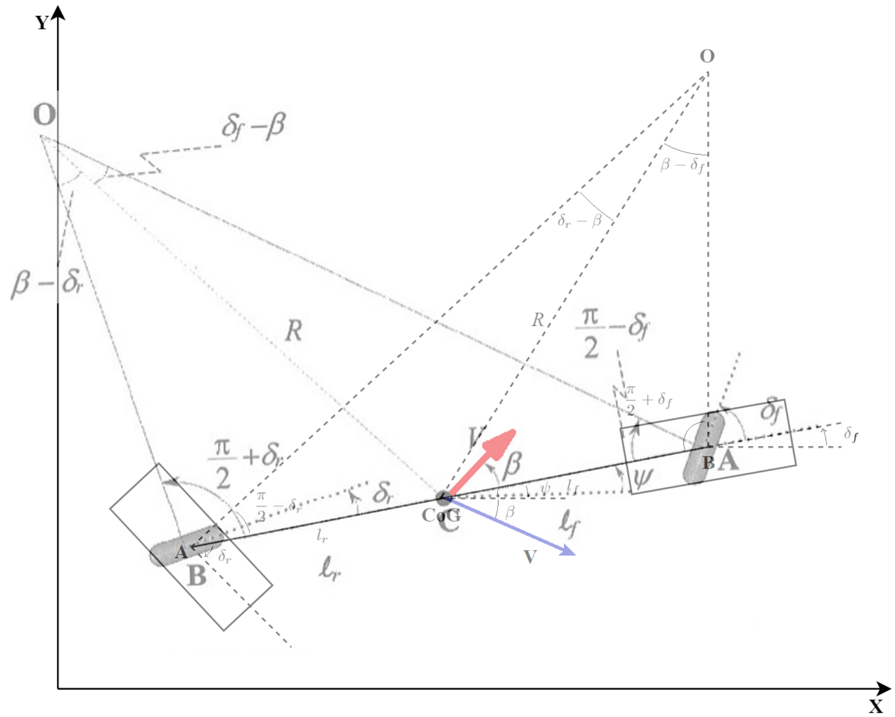

Figure 2: Kinematics of lateral vehicle motion for rear steering bicycle model figure 1 overlaid on top of Rajamani's front steering bicycle model [2, p. 21]. Blue vector is a rear steering vehicle. Red Vector is a front steering vehicle.

By overlaying the rear and front steering model's some immediate observations can be made figure 2. The vehicles stacked on top of one another are both rotating in a counter-clockwise direction around their point **O**s respectively, however the velocity vectors are pointing in opposite directions, toward the turning point of the radius of the turn for front steering, and out away from the turning point of the radius of the turn for rear steering.

The turning point of the radius of the turn for the front steering is behind the **CoG** while for the rear steering is in the front. This is directly from the result of having rear steering. The rear steering model's radius vectors are a mirror image of the front steering model, yet behaves so differently due to, the velocity vectors that are pointed in completely different directions drastically changing the behavior of the car.

Viewing the difference between the velocity vectors of the car it shows how the front steering vehicle is pulling the car in the direction of the steered wheels, but the rear steering vehicle is pushing the non-turning tires. This pushing versus pulling can be thought of as a classic red wagon. When the wagon is pulled it will follow the direction of the handle. However pushing the wagon backwards is much harder and difficult to control, however not impossible.

Sine rule for triangle **OACoG**

$$\frac{\sin(\delta_r - \beta)}{l_r} = \frac{\sin(\frac{\pi}{2} - \delta_r)}{R} \tag{2.1}$$

$$\frac{\sin(\delta_r)\cos(\beta) - \sin(\beta)\cos(\delta_r)}{l_r} = \frac{\cos(\delta_r)}{R} \tag{2.2}$$

multiply Eq. (2.2) by $\frac{l_r}{\cos(\delta_r}$

$$\tan(\delta_r)\cos(\beta) - \sin(\beta) = \frac{l_r}{R} \tag{2.3}$$

Sine rule for triangle **OBCoG**

$$\frac{\sin(\beta - \delta_f)}{l_f} = \frac{\sin(\frac{\pi}{2} + \delta_f)}{R} \tag{2.4}$$

$$\frac{\cos(\delta_f)\sin(\beta) - \cos(\beta)\sin(\delta_f)}{l_f} = \frac{\cos(\delta_f)}{R} \tag{2.5}$$

multiply Eq. (2.5) by $\frac{l_f}{\cos(\delta_f}$

$$\sin(\beta) - \tan(\delta_f)\cos(\beta) = \frac{l_f}{R} \tag{2.6}$$

add Eq. (2.3) to Eq. (2.6)

$$(\tan(\delta_r) - \tan(\delta_f))\cos(\beta) = \frac{l_f + l_r}{R} \tag{2.7}$$

$$\dot{\psi} = \frac{V}{R} \tag{2.8}$$

$$\dot{\psi} = \frac{V\cos(\beta)}{l_f + l_r}(\tan(\delta_f) - \tan(\delta_r)) \tag{2.9}$$

$\delta_f$ is set to 0 since it is a rear steered vehicle

$$\tag{2.10}$$

The equations of motion for a rear steering bicycle model in terms of $\dot{X}$, $\dot{Y}$, and $\dot{\psi}$ can be written as:

$$\dot{X} = V\cos(\psi + \beta) \tag{3.1}$$

$$\dot{Y} = V\sin(\psi + \beta) \tag{3.2}$$

$$\dot{\psi} = \frac{V\cos(\beta)}{l_f + l_r}(\tan(\delta_r)) \tag{3.3}$$

$$\beta = \tan^{-1}(\frac{l_f\delta_r}{l_f + l_r}) \tag{3.4}$$

Rajamani's equations of motion [2, p. 27] for a front steering bicycle model in terms of $\dot{X}$, $\dot{Y}$, and $\dot{\psi}$ are the following:

$$\dot{X} = V\cos(\psi + \beta) \tag{4.1}$$

$$\dot{Y} = V\sin(\psi + \beta) \tag{4.2}$$

$$\dot{\psi} = \frac{V\cos(\beta)}{l_f + l_r}(\tan(\delta_f)) \tag{4.3}$$

$$\beta = \tan^{-1}(\frac{l_r\delta_f}{l_f + l_r}) \tag{4.4}$$

The equations of motion for a rear steered bicycle model is the same as of a front steered bicycle model that is changed from $\delta_f$ to $\delta_r$. This makes sense since the kinematic bicycle model was written to work for both forward and reverse dynamics. Being able to verify and see that this is the case solidifies the solution of the bicycle problem.

# 3 Dynamic Rear Steering Bicycle Model

The dynamic model however can not be used for reversing and must be changed to represent the reversing dynamics of a vehicle. This dynamic bicycle model only represents two degrees of freedom, rotational and lateral positioning relative to a road. This can be used and implemented into a control algorithm.

The sum of the lumped front and rear tire forces is equal to the amount of lateral force exerted on the road plus the banking force (centripetal force).

$$\sum F_y = ma_y = F_{yfront} + F_{yrear} + F_{bank} \tag{5.1}$$

The acceleration in the y direction is equal to the second derivative of position in $y$ and the centripetal acceleration of the car rotating.

$$a_y = \ddot{y} + V_x\psi \tag{5.2}$$

Moment balance around z-axis for bicycle model [2, p. 29].

$$I_z\ddot{\psi} = l_f F_{yfront} - l_r F_{yrear} \tag{5.3}$$

The rear tire is now the front tire due to the rear steering. For small slip-angles the lateral tire force is proportional to the tire force [2, p. 29].

$$a_{front} = -\theta_{vfront} \tag{5.4}$$

$$a_{rear} = \delta - \theta_{vrear} \tag{5.5}$$

Banking force equation [2, p. 32].

$$F_{bank} = mg\sin(\phi) \tag{5.6}$$

Force Exerted on the tires [2, p. 29].

$$F_{yrear} = 2c_{\alpha r}(\delta - \theta_{vrear}) \tag{5.7}$$

$$F_{yfront} = 2c_{\alpha r}(-\theta_{vfront}) \tag{5.8}$$

Using a small angle approximation the tire force angle is found [2, p. 29].

$$\tan(\theta_{vfront}) \approx \theta_{vfront} = \frac{\dot{y} - l_{front}\dot{\psi}}{V_x} \tag{5.9}$$

$$\tan(\theta_{vrear}) \approx \theta_{vrear} = \frac{\dot{y} + l_{rear}\dot{\psi}}{V_x} \tag{5.10}$$

Substitute Eqs. 5.2, 5.4, 5.5, and 5.6.

$$m(\ddot{y} + V_x\psi) = 2c_{\alpha r}(\delta - \theta_{vrear}) + 2c_{\alpha r}(-\theta_{vfront}) + mg\sin(\phi) \tag{5.11}$$

$$\ddot{y} = \frac{2c_{\alpha r}(-\frac{\dot{y}-l_{front}\dot{\psi}}{V_x})}{m} + \frac{2c_{\alpha r}(\delta - \frac{\dot{y}+l_{rear}\dot{\psi}}{V_x})}{m} + \frac{mg\sin(\phi)}{m} - V_x\dot{\psi} \tag{5.12}$$

$$\ddot{\psi} = \frac{l_f 2c_{\alpha f}(-\frac{\dot{y}-l_{front}\dot{\psi}}{V_x}) - l_r 2c_{\alpha r}(\delta - \frac{\dot{y}+l_{rear}\dot{\psi}}{V_x})}{I_z} \tag{5.13}$$

A state space model can be written for a dynamic rear steering bicycle model using Eqs. 5.12 and 5.13.

$$\frac{d}{dt}\begin{bmatrix} y \\ \dot{y} \\ \psi \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{2c_{\alpha f}+2c_{\alpha r}}{mV_x} & 0 & -V_x - \frac{2c_{\alpha r}l_r - 2c_{\alpha f}l_f}{mV_x} \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{2l_f c_{\alpha f} - 2l_r c_{\alpha r}}{I_z V_x} & 0 & \frac{2l_f^2 c_{\alpha f} + 2l_r^2 c_{\alpha r}}{I_z V_x} \end{bmatrix} \begin{bmatrix} y \\ \dot{y} \\ \psi \\ \dot{\psi} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{2c_{\alpha r}}{m} \\ 0 \\ -\frac{l_r 2c_{\alpha r}}{I_z} \end{bmatrix} \delta + \begin{bmatrix} 0 \\ g \\ 0 \\ 0 \end{bmatrix} \sin(\phi) \tag{6.1}$$

The state space model for a dynamic front steering bicycle model by Rajamani [2, p. 31].

$$\frac{d}{dt}\begin{bmatrix} y \\ \dot{y} \\ \psi \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{2c_{\alpha f}+2c_{\alpha r}}{mV_x} & 0 & -V_x - \frac{2c_{\alpha f}l_f - 2c_{\alpha r}l_r}{mV_x} \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{2l_f c_{\alpha f} - 2l_r c_{\alpha r}}{I_z V_x} & 0 & -\frac{2l_f^2 c_{\alpha f} + 2l_r^2 c_{\alpha r}}{I_z V_x} \end{bmatrix} \begin{bmatrix} y \\ \dot{y} \\ \psi \\ \dot{\psi} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{2c_{\alpha f}}{m} \\ 0 \\ \frac{l_f 2c_{\alpha f}}{I_z} \end{bmatrix} \delta \tag{7.1}$$

The resulting differences between the dynamic rear steering model and front steering model is from changing the front to rear steering steering angle $\delta$. The differences in the state space models are the signs affecting the terms for $\ddot{\psi}$. This shows that by switching $\delta_f$ to $\delta_r$ allows for the forward steered bicycle model to represent a rear bicycle model. Since the steering is now in the rear the terms in the $\delta$ matrix are now for the rear tire.

# 4 Conclusion

The implementation of the kinematic model resulted in showing that the front steering and rear steering kinematic model yield the same result, this means that the kinematic model can be used for both forwards and reversing vehicle dynamics. The state space model however changes due to the nature of the calculations made for front steering and is required to be adjusted for rear steering vehicles. The rear steering state space model can be used for implementing robust control algorithms. Using this derivation, quick driving in reverse would hopefully be possible and be implemented in the near future for autonomous driving. With further testing, the accuracy of this dynamic rear steering bicycle model can be tested and see if it is comparable to the degree of accuracy to a forward moving bicycle model.

## 5 Nomenclature

| | |
|---|---|
| $F_y$ | lateral force |
| $Fy_f$ | lateral tire force front wheels |
| $Fy_r$ | lateral tire force rear wheels |
| $V_x$ | logitudinal speed |
| $l_f$ | logitudinal distance from CoG to front axle |
| $l_r$ | logitudinal distance from CoG to rear axle |
| $\phi$ | bank angle |
| $c_{\alpha r}$ | cornering stiffness rear tire |
| $c_{\alpha f}$ | cornering stiffness rear tire |
| $\theta_{vr}$ | rear tire velocity angle |
| $\theta_{vf}$ | front tire velocity angle |
| $I_z$ | moment of intertia around CoG in Z axis |
| $m$ | mass of vehicle |
| $\psi$ | yaw angle |
| $\beta$ | vehicle slip angle |
| $\delta$ | steering angle |
| $\delta_f$ | average front wheel steering angle |
| $\delta_r$ | average rear wheel steering angle |

## References


[1] L. Alexander, C. Zhu, and R. Rajamani*. Lateral control of a backward driven front-steering vehicle. *Control Engineering Practice*, 11:531–540, 2003.

[2] Rajesh Rajamani. *Vehicle Dynamics and Control.* Springer Science+Business Media, 2006.